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1988 J. Phys. A: Math. Gen. 21 1913

(http://iopscience.iop.org/0305-4470/21/8/027)

J. Phys. A: Math. Gen. **21** (1988) 1913–1915. Printed in the UK

COMMENT

An example of a Miura map: the shallow-water wave equations

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Received 10 September 1987, in final form 14 January 1988

Abstract. The linear Hamiltonian structure of 'modified' shallow-water wave equations goes to the non-linear and second Hamiltonian structure of the bi-Hamiltonian shallow-water wave equations. A similar situation must exist for gas dynamics according to a result of Dubrovin and Novikov.

1. Introduction

Miura (1968) proposed a remarkable non-linear map taking solutions of the modified Korteweg–de Vries (MKdV) equation to those of the Korteweg–de Vries (KdV) equation itself. Wadati and Sogo (1983) found a Miura map relating one of the many modified non-linear Schrödinger equations (MNLS) to the original NLS (see Clarkson and Cosgrave (1987) for a catalogue of MNLS). Finally, Kupershmidt (1985), as part of a profound study of the Hamiltonian structures of dispersive water waves, presented a general form of all Miura maps.

In Verosky (1987), the relationship between the NLS and the shallow-water wave equations (SWW)

$$v_t + vv_x + \sigma_x = 0 \qquad \sigma_t + \sigma v_x + v\sigma_x = 0$$

and their bi-Hamiltonian structures was explored. The concept of bi-Hamiltonian structure was first published by Magri (1978) as a way of illuminating the infinite sequence of conserved densities and symmetries of the various 'completely integrable' equations such as KdV, MKdV, NLS, etc. The SWW are a sort of classical limit of NLS, but in practice this amounts to dropping higher-order terms after a Madelung (1927) type change of variables. It turns out that the Miura transformation for NLS has its analogue in SWW. There is a corresponding modified SWW (MSWW) and the relationship of the bi-Hamiltonian structures is analogous to the situation for KdV and MKdV. This relationship is a special case of the general Miura maps in Kupershmidt's work, but it is a concrete and simple example of the phenomenon in the case of matrix Hamiltonian operators.

2. The Miura map for SWW

The change of variables is given by

$$\sigma = \varepsilon\rho u - \varepsilon^2\rho^2 \qquad v = u.$$

The resulting equations are

$$u_t + uu_x + (\varepsilon\rho u - \varepsilon^2\rho^2)_x = 0 \qquad \rho_t + (\rho u)_x - (\tfrac{1}{2}\varepsilon\rho^2)_x = 0$$

and shall be called MSWW. These are the 'classical limit' of one of the MNLS to be found in Clarkson and Cosgrave (1987) or Kundu (1984). The pressure $\varepsilon\rho u - \varepsilon^2\rho^2$ depends on momentum ρu and there is a new term $-\frac{1}{2}\varepsilon\rho^2$ in the density flux. The linear Hamiltonian structure

$$\begin{pmatrix} u \\ \rho \end{pmatrix}_t = -\begin{pmatrix} 2\varepsilon\mathrm{D} & \mathrm{D} \\ \mathrm{D} & 0 \end{pmatrix}\begin{pmatrix} \partial_u \\ \partial_\rho \end{pmatrix}(\tfrac{1}{2}\rho u^2 - \tfrac{1}{2}\varepsilon\rho^2 u)$$

of these modified equations must go over to the non-linear structure of SWW whose existence was first discovered by Nutku (1987). The Hamiltonian structure in the old variables is then

$$\begin{pmatrix} v \\ \sigma \end{pmatrix}_t = -\begin{pmatrix} 1 & 0 \\ \varepsilon\rho & \varepsilon u - 2\varepsilon^2\rho \end{pmatrix}\begin{pmatrix} 2\varepsilon\mathrm{D} & \mathrm{D} \\ \mathrm{D} & 0 \end{pmatrix}\begin{pmatrix} 1 & \varepsilon\rho \\ 0 & \varepsilon u - 2\varepsilon^2\rho \end{pmatrix}\begin{pmatrix} \partial_v \\ \partial_\sigma \end{pmatrix}\left[\frac{\sigma v}{2\varepsilon}\right]$$

where the Jacobian of the change of variables enters in an obvious way. After some surprising algebra in the matrix operators, Nutku's celebrated non-linear Hamiltonian structure is obtained:

$$\begin{pmatrix} v \\ \sigma \end{pmatrix}_t = -\begin{pmatrix} 2\mathrm{D} & \frac{1}{2}\mathrm{D}v \\ \frac{1}{2}v\mathrm{D} & \frac{1}{2}(\sigma\mathrm{D} + \mathrm{D}\sigma) \end{pmatrix}\begin{pmatrix} \partial_v \\ \partial_\sigma \end{pmatrix}[\sigma v].$$

Of course, a non-linear Hamiltonian structure by itself is not amazing, but if the same system of equations also has a linear structure or a second non-linear one then this is the case of a bi-Hamiltonian structure. The linear structure for SWWE is the simple

$$\begin{pmatrix} v \\ \sigma \end{pmatrix}_t = -\begin{pmatrix} 0 & \mathrm{D} \\ \mathrm{D} & 0 \end{pmatrix}\begin{pmatrix} \partial_v \\ \partial_\sigma \end{pmatrix}(\tfrac{1}{2}\sigma v^2 + \tfrac{1}{2}\sigma^2).$$

3. Generalisation to other first-order systems

A result of Dubrovin and Novikov (1983) is that any first-order Hamiltonian system

$$\begin{pmatrix} p \\ q \end{pmatrix}_t = J\begin{pmatrix} \partial_p \\ \partial_q \end{pmatrix} H(p, q)$$

where J is a first-order, possibly non-linear, matrix Hamiltonian operator, can be reduced to a system with a linear Hamiltonian structure by a change of variables $p = f(r, s)$, $q = g(r, s)$. The new system is

$$\begin{pmatrix} r \\ s \end{pmatrix}_t = \begin{pmatrix} f_p & f_q \\ g_p & g_q \end{pmatrix} J \begin{pmatrix} f_p & g_p \\ f_q & g_q \end{pmatrix} H(f(r, s), g(r, s))$$

where the operator

$$\begin{pmatrix} f_p & f_q \\ g_p & g_q \end{pmatrix} J \begin{pmatrix} f_p & g_p \\ f_q & g_q \end{pmatrix}$$

is now linear. If the original pq system also had a linear Hamiltonian structure and thus a bi-Hamiltonian structure then the transformation $r = h(p, q)$, $s = k(p, q)$ would be a 'Miura map'. The point of this is that bi-Hamiltonian structures and Miura maps always occur at exactly the same time for first-order systems. The number of dependent variables can be more than two since they can be so in the Dubrovin–Novikov result.

According to Nutku's work, the equations of gas dynamics

$$u_t + uu_x + v^{\gamma-2} v_x = 0$$

$$v_t + vu_x + uv_x = 0$$

have bi-Hamiltonian structures and must thus have Miura map and 'modified' gas dynamic analogues, when the Dubrovin–Novikov transformations are found.

Acknowledgments

I would like to thank Yavuz Nutku and Peter Olver for many enlightening discussions which provided the mental framework making this work possible. This work was partially supported by NSF grant no DMS 86-02004, and by the UK Science and Engineering Research Council. Finally, I am pleased to thank the referees for several helpful comments.

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